Closing Today: HW_5C (7.3)
Closing *Thurs*: HW_6A, 6B (7.4, 7.5)

7.4 Partial Fractions

Motivation: We will learn to break-up fractions like:

$$\frac{x^3+4x-4}{x^2(x^2+4)}=\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^2+4}$$

Entry Task 1: If I tell you the above relationship is true, then integrate

$$\int \frac{x^3+4x-4}{x^2(x^2+4)}\,dx$$

$$= \int \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^2+4}\,dx$$

$$= \ln|x| + \frac{1}{x} + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

NOTE: $\int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{\frac{x^2}{4}+1}\,dx$

$$= \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx \qquad t=\tfrac{1}{2}x,\quad dt=\tfrac{1}{2}dx,\quad 2dt=dx$$

$$= \frac{1}{4}\int \frac{1}{t^2+1}\,2dt$$

$$= \frac{1}{2}\tan^{-1}(t)+C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

Entry Task 2: Do you know **long-division**?
Divide in order to fill in the question marks:

$$\frac{576}{11} = ? + \frac{?}{11}$$

$$\begin{array}{r} 52 \\ 11\,\overline{)\,576} \\ -55 \\ \hline 26 \\ -22 \\ \hline 4 \end{array}$$

$$\Rightarrow \frac{576}{11} = 52 + \frac{4}{11}$$

Partial Fraction Decomposition

Step 0: Is the fraction *reduced*?

reduced - highest power on top smaller than the highest power on bottom.

If yes, move to step 1.

If not, divide, then move to step 1.

Example:

$$\int \frac{x^2+x}{x+3}\,dx$$

HIGHEST NUM. POWER = 2

HIGHEST DEN. POWER = 1

$2 \geq 1 \Rightarrow$ DIVIDE!

$$\begin{array}{r|l} & x - 2 \\ \hline x+3 & x^2 + x \\ & -(x^2+3x) \\ \hline & -2x \\ & -(-2x-6) \\ \hline & 6 \end{array}$$

$$\Rightarrow \frac{x^2+x}{x+3} = x - 2 + \frac{6}{x+3}$$

CHECK $\frac{(x-2)(x+3)}{x+3} + \frac{6}{x+3}$

$\frac{x^2+x-6+6}{x+3}$ ✓

$$= \int x - 2 + \frac{6}{x+3}\,dx$$

$$= \boxed{\frac{1}{2}x^2 - 2x + 6\ln|x+3| + C}$$

CHECK $\frac{d}{dx} \to x - 2 + \frac{6}{x+3}$ ✓

Partial Fractions Method Summary

Step 0: Reduce (if needed), see last page.

Step 1: Factor Denominator.
Write out decomposition below:

i) Distinct Linear:

$$\frac{x^2-3}{x(x-1)(x+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+4}$$

ii) Repeated Linear:

$$\frac{5+2x}{(x+3)(x-2)^3} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

iii) Irreducible Quadratic:

$$\frac{4x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

Step 2: Solve for A, B, C ….

Step 3: Integrate

All the integrals in this section look like these:

$$\int \frac{1}{2x+5}\,dx = \frac{1}{2}\ln|2x+5| + C$$

$$\int \frac{1}{(x-4)^2}\,dx = -\frac{1}{x-4} + C$$

$$\int \frac{1}{(x+7)^3}\,dx = -\frac{1}{2}\frac{1}{(x+7)^2} + C$$

$$\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\ln|x^2+9| + C$$

The method uses algebra to rewrite ***any*** rational function as a sum of the integrals like those above.

Example:

$$\int \frac{x+1}{x^2-4}\,dx$$

TOP POWER = 1

BOT POWER = 2

$1 < 2 \Rightarrow$ ~~[illegible]~~ REDUCED ✓

DON'T NEED TO DIVIDE

FACTOR!

$$\frac{x+1}{(x+2)(x-2)} \overset{?}{=} \frac{A}{x+2} + \frac{B}{x-2}$$

$$\Rightarrow \quad x+1 = A(x-2) + B(x+2)$$

$$\Rightarrow \quad x+1 = Ax - 2A + Bx + 2B$$

$$\underbrace{x+1} = \underbrace{(A+B)x + (-2A+2B)}$$

WANT ALWAYS THE SAME!

MATCH COEFFICIENTS! $\Rightarrow \quad A+B = 1 \Rightarrow B = 1 - A$

and $-2A + 2B = 1 \Rightarrow -2A + 2(1-A) = 1$

$\Rightarrow \quad -4A = -1 \Rightarrow A = 1/4$

$B = 1 - A = 3/4$

SHORTCUT!

LET $x = 2 \Rightarrow \quad 3 = A(2-2) + B(2+2)$

$\Rightarrow B = 3/4$

$x = -2 \Rightarrow \quad -1 = A(-2-2) + B(-2+2)$

$\Rightarrow A = 1/4$

$$\int \frac{1/4}{x+2} + \frac{3/4}{x-2}\,dx$$

$$\boxed{\frac{1}{4}\ln|x+2| + \frac{3}{4}\ln|x-2| + C}$$

Example:

$$\int \frac{x+1}{x^3+3x^2}\,dx$$

REDUCED ✓

FACTOR DENOM

$$\frac{x+1}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$$

REPEATED

$$\Rightarrow \quad x+1 = Ax(x+3) + B(x+3) + Cx^2$$

$x=0 \Rightarrow 1 = A(0) + B(3) + C(0)$

$\Rightarrow \boxed{B = 1/3}$

$x=-3 \Rightarrow -2 = A(0) + B(0) + C(9)$

$\Rightarrow \boxed{C = -2/9}$

$x+1 = Ax^2 + 3Ax + Bx + 3B + Cx^2$

$x+1 = (A+C)x^2 + (3A+B)x + 3B$

$\Downarrow$

$A+C = 0 \Rightarrow A = -C$

$3A+B = 1$

$3B = 1$

$\boxed{A = -(-2/9) = 2/9}$

$$\int \frac{x+1}{x^2(x+3)}\,dx$$

$$= \int \frac{2/9}{x} + \frac{1/3}{x^2} + \frac{-2/9}{x+3}\,dx$$

$$= \frac{2}{9}\ln|x| - \frac{1}{3}\frac{1}{x} - \frac{2}{9}\ln|x+3| + C$$

ASIDE:

$$\frac{2}{9}\left(\ln|x| - \ln|x+3|\right) = \frac{2}{9}\ln\left|\frac{x}{x+3}\right|$$

Example:

$$\int \frac{x^2 - x + 6}{x^3 + 3x}\,dx$$

REDUCED ✓

FACTOR DENOM

$$\frac{x^2 - x + 6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3}$$

$$\int \frac{x^2 - x + 6}{x(x^2+3)}\,dx$$

$$\Rightarrow x^2 - x + 6 = A(x^2+3) + (Bx + C)x$$

$x = 0 \Rightarrow 6 = A(3) + (B(0) + C)(0)$

$\Rightarrow A = 2$

$$x^2 - x + 6 = Ax^2 + 3A + Bx^2 + Cx$$

$\Rightarrow A + B = 1 \Rightarrow B = 1 - A = -1$

$C = -1$

$3A = 6$ ✓

→ SPLIT UP!

$$= \int \frac{2}{x} + \frac{-x-1}{x^2+3}\,dx$$

$$= \int \frac{2}{x}\,dx - \int \frac{x}{x^2+3} - \int \frac{1}{x^2+3}\,dx$$

$$= 2\ln|x| - \int \frac{x}{u}\frac{1}{2x}\,du - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$$= 2\ln|x| - \frac{1}{2}\ln|u| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$u = x^2 + 3$

$du = 2x\,dx$

$\frac{1}{2x}du = dx$

$$= \boxed{2\ln|x) - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$$

Example:

$$\int \frac{x}{x^2+4x+5}\,dx$$

REDUCED ✓

FACTOR DENOM???

$x^2+4x+5=0$ HAS NO REAL SOL'NS

$x = \dfrac{-4 \pm \sqrt{16-4(5)}}{2}$

-4 under radical

IRREDUCIBLE

$\Downarrow$

COMPLETE THE SQUARE!

$$x^2+\underbrace{4x}_{\text{HALF}\,\to\, 2}+\underbrace{4}_{\text{SQUARED}}-4+5=(x+2)^2+1$$

$$\int \frac{x}{(x+2)^2+1}\,dx$$

to simplify

$t = x+2 \rightarrow x = t-2$

$dt = dx$

$$\int \frac{t-2}{t^2+1}\,dt$$

NOW SPLIT UP!

CE AST AGE

$$\int \frac{t}{t^2+1}\,dt - \int \frac{2}{t^2+1}\,dt$$

$$= \frac{1}{2}\ln(t^2+1) - 2\tan^{-1}(t) + C$$

$$= \boxed{\frac{1}{2}\ln((x+2)^2+1) - 2\tan^{-1}(x+2) + C}$$

How to integrate

A. Look for simplifications/substitutions

B. Products/Logs/Inverse Trig → BY PARTS
   Sin/Cos/Tan/Sec combos → TRIG
   Quadratic (under a radical) → TRIG SUB
   Rational Function → PART. FRAC.

C. If nothing seems to work, substitution.
   (u = inside, u = $\sqrt{\ }$, u = trig, u = e^x)

Examples of substitution:

1. $\int e^{\sqrt{x}}dx$

$t=\sqrt{x} \Rightarrow t^2=x$

$2t\,dt=dx$

$\int e^t\, 2t\,dt$

$\int 2te^t\,dt$

NOW BY PARTS!

2. $\int \frac{3}{x-2\sqrt{x}}dx$

$t=\sqrt{x} \Rightarrow t^2=x$

$2t\,dt=dx$

$\int \frac{3}{t^2-2t}\,2t\,dt$

$= \int \frac{6t}{t^2-2t}\,dt$

NOW FACTOR DENOMINATOR!

3. $\int \frac{\cos(x)}{4-\sin^2(x)}dx$

$t=\sin(x)$

$dt=\cos(x)dx$

$\int \frac{1}{4-t^2}\,dt$

NOW FACTOR DENOMINATOR!

4. $\int e^x\cos(e^x)\sin^3(e^x)\,dx$

$t=e^x$

$dt=e^x dx$

$\int \cos(t)\sin^3(t)\,dt$

NOW "PULL-OUT" COS(t) AND $u=\sin(t)$!

How would you *start* these?

1. $\int \tan^3(x)\sec(x)\,dx = \int \tan^2(x)\sec(x)\tan(x)\,dx$ $\qquad u=\sec(x)$

TRIG! $\qquad = \int (\sec^2(x)-1)\sec(x)\tan(x)\,dx$ $\qquad du=\sec(x)\tan(x)\,dx$

---

2. $\int x^2\ln(x)\,dx = \frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^2\,dx$ $\qquad u=\ln(x) \quad dv=x^2\,dx$

BY PARTS! $\qquad = \frac{1}{3}x^3\ln|x| - \frac{1}{9}x^3 + C$ $\qquad du=\frac{1}{x}\,dx \quad v=\frac{1}{3}x^3$

---

3 $\int x\sqrt{5-x^2}\,dx = \int x\sqrt{u}\,\frac{1}{-2x}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}+C$ $\qquad u=5-x^2 \leftarrow$ EASIER $\qquad x=\sqrt{5}\sin(\theta)$

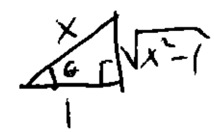
SUB. (OR TRIG SUB) $\qquad = -\frac{1}{3}(5-x^2)^{3/2}+C$ $\qquad du=-2x\,dx$ $\quad$ OR

$-\frac{1}{2x}\,du = dx$

---

4. $\int \frac{\sqrt{x^2-1}}{x^2}\,dx = \int \frac{\sqrt{\sec^2\theta-1}}{\sec^2\theta}\sec\theta\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$ OR $\qquad x=\sec\theta$

TRIG SUB $\qquad \int \frac{\sec^2\theta-1}{\sec\theta}\,d\theta = \int \frac{\sin^2\theta}{\cos\theta}\,d\theta$ $\qquad dx=\sec\theta\tan\theta\,d\theta$



---

5. $\int \frac{x^2+1}{x^2-2x-3}\,dx = \int 1 + \frac{2x+4}{x^2-2x-3}\,dx$

$$x^2-2x-3\,\overline{)\,x^2+1\,}\quad 1 \qquad \frac{-(x^2-2x-3)}{2x+4}$$

DIVIDE, THEN FACTOR DENOM $\qquad \frac{2x+4}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

---

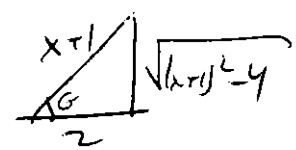
6. $\int x\tan^{-1}(x)\,dx = \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx$ $\qquad u=\tan^{-1}(x) \quad dv=x\,dx$

BY PARTS $\qquad \uparrow$ DIVIDE!! $\qquad du=\frac{1}{x^2+1}\,dx \quad v=\frac{1}{2}x^2$

---

7. $\int \frac{dx}{\sqrt{4x^2+8x-12}}\,dx$ $\qquad \sqrt{4(x^2+2x+1-1-3)} = 2\sqrt{(x+1)^2-4}$

$\searrow \int \frac{1}{2\sqrt{4\sec^2\theta-4}}\,2\sec\theta\tan\theta\,d\theta$ $\qquad x+1=2\sec\theta$



COMPLETE SQUARE & TRIG SUB! $\qquad = \frac{1}{2}\int \frac{1}{\cancel{2}\cancel{\tan\theta}}\,\cancel{2}\sec\theta\,\cancel{\tan\theta}\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta|+C$